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# An asymptotic analysis of two-phase fluid mixing*

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**Abstract.** We study the motion of the slightly compressible multi-phase flow model proposed by Chen, Glimm, Sharp and Zhang. The interface velocity and constitutive law are analyzed by derivation of the exact quantity. Using singular perturbation theory, a formal asymptotic expansion is derived for the solution of the compressible equations. An asymptotic analysis in the incompressible limit, for slightly compressible flows supplies important new information to resolve nonuniqueness of the pressure difference between the two fluid species of the incompressible flow equations.

**Keywords:** multiphase flow, asymptotic expansions, incompressible two phase fluids.

## 1 Introduction

We study the motion of the slightly compressible chunk mix multi-phase flow model proposed by Chen et al [1]. The multi-phase flow model provides an averaged, or coarse gained, description of the mixing layer formed when an unstable interface between two fluids is driven by acceleration. New closures of the two pressure model of multi-phase flow have been proposed recently [4, 8, 9, 10, 11]. The physical basis for the model and algorithms for numerical solutions of the model equations have been examined [2, 3, 6, 7]. A closed form solution was introduced in the incompressible case [11]. We derive an exact expression for the interface velocity by manipulation of the governing equations. The formulation implies that the velocity constitutive law is not a free

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parameter. We present a zero parameter model, with the velocity constitutive law given implicitly in terms of the solution.

The equations of compressible multi-phase flow in appropriate nondimensional form are a nonlinear hyperbolic system depending on a large parameter $\lambda$, the reciprocal of the Mach number. The incompressible limit of the compressible multi-phase flow equations is a time-singular and layer-type problem which requires advanced techniques in asymptotics. We discuss here the outer limiting behavior of the solutions of the compressible multi-phase equations as $\lambda \to \infty$. Using singular perturbation theory, we derive formal asymptotic expansions of the solutions of the compressible equations, describing the outer limiting problem, which are uniformly valid in space. A necessary and sufficient condition for convergence of compressible pressures through the second order of the asymptotic expansion to the incompressible pressures is derived. An additional degree of freedom exists in the pressures for the incompressible limit. This condition fixes a degree of freedom related to the relative compressibility of the two fluids in the incompressible pressures.

## 1.1 The Flow Equations

The fluids are distinguished by a subscript $k$, where $k = 1$ and $k = 2$ denote the light and heavy fluids, respectively, and the primed index $k'$ denotes the fluid complementary to fluid $k$, *i.e.*, $k' = 3 - k$. The dependent variables are $\beta_k$, $\rho_k$, $v_k$, and $p_k$, which denote, respectively, the volume fraction, density, velocity, and pressure of fluid (phase) $k$. We allow here the possibility that an externally imposed acceleration $g = g(t) > 0$ is time dependent. The equations of motion are

$$\frac{\partial \beta_k}{\partial t} + v^* \frac{\partial \beta_k}{\partial z} = 0\,, \tag{1.1.1}$$

$$\beta_k\left(\frac{\partial \rho_k}{\partial t} + v_k\frac{\partial \rho_k}{\partial z}\right) + \beta_k\rho_k\frac{\partial v_k}{\partial z} + \rho_k(v_k - v^*)\frac{\partial \beta_k}{\partial z} = 0\,, \tag{1.1.2}$$

$$\beta_k\rho_k\left(\frac{\partial v_k}{\partial t} + v_k\frac{\partial v_k}{\partial z}\right) + \beta_k\frac{\partial p_k}{\partial z} + (p_k - p^*)\frac{\partial \beta_k}{\partial z} = \beta_k\rho_k g(t), \tag{1.1.3}$$

together with the constraint

$$\beta_1 + \beta_2 = 1\,, \tag{1.1.4}$$

where $p_k = A_k\rho_k^{\gamma_k}$, $\gamma_k > 1$, $A_k$ is the entropy of the fluid and assumed to be constant within each fluid but $A_1 \neq A_2$, and $\frac{dp_k(\rho_k)}{d\rho_k} > 0$ for $\rho_k > 0$. The quantities $v^*$ and $p^*$ represent averages of microscopic quantities. In [8, 10] a general model for the average interfacial quantities $q^*$, $q = v, p$, was proposed and examined against DNS data of Rayleigh-Taylor instabilities. We denote $Z_k = Z_k(t)$ as the position of the mixing zone edge $k$, defined as the location of vanishing $\beta_k$ and $V_k = \dot{Z}_k$ as the velocity of the edge $k$. At edge $k$, the following boundary data holds

$$v_k = V_k(t) \quad \text{at } z = Z_k(t)\,. \tag{1.1.5}$$

An exact expression for $v^*$ in Eq. (1.1.1) is derived from the compressible continuity equations

$$\frac{\partial(\beta_k\rho_k)}{\partial t} + \frac{\partial(\beta_k\rho_k v_k)}{\partial z} = 0\,. \tag{1.1.6}$$

Eq. (1.1.6), with the spatial derivatives expanded, may be regarded as two equations ($k$ =1, 2) in the two unknown quantities $\partial\beta_1/\partial t$ and $\partial\beta_1/\partial z$. These equations are easily solved, and the result when substituted into Eq. (1.1.1) yields

$$\begin{aligned} v^* &= \frac{\beta_1\left(\frac{\partial v_1}{\partial z} + \frac{1}{\rho_1}\frac{D_1\rho_1}{Dt}\right)v_2 + \beta_2\left(\frac{\partial v_2}{\partial z} + \frac{1}{\rho_2}\frac{D_2\rho_2}{Dt}\right)v_1}{\beta_1\left(\frac{\partial v_1}{\partial z} + \frac{1}{\rho_1}\frac{D_1\rho_1}{Dt}\right) + \beta_2\left(\frac{\partial v_2}{\partial z} + \frac{1}{\rho_2}\frac{D_2\rho_2}{Dt}\right)} \\ &= \frac{\beta_1 v_2 + d_1^v\beta_2 v_1}{\beta_1 + d_1^v\beta_2}\,, \end{aligned} \tag{1.1.7}$$

where the phase $k$ convective derivative is denoted $D_k/Dt \equiv \partial/\partial t + v_k\partial/\partial z$ and the coefficient

$$d_k^v = \frac{\dfrac{\partial v_{k'}}{\partial z} + \dfrac{1}{\rho_{k'}}\dfrac{D_{k'}\rho_{k'}}{Dt}}{\dfrac{\partial v_k}{\partial z} + \dfrac{1}{\rho_k}\dfrac{D_k\rho_k}{Dt}} = \frac{\dfrac{1}{\beta_{k'}}\dfrac{D_{k'}\beta_{k'}}{Dt}}{\dfrac{1}{\beta_k}\dfrac{D_k\beta_k}{Dt}} \tag{1.1.8}$$

is a ratio of logarithmic rates of volume creation for the two phases. Refer to [16]. The constitutive factor $d_k^v$ is given in terms of the solution and it thus is not a free parameter. It is obtained by manipulation of the original unclosed equations and it thus has no content which goes beyond these equations. These equations are not sufficient to determine any of the primitive variables. Any new relation involving the primitive variables is in principle admissible as a closure.

## 1.2 Closure

From (1.1.7), the interface velocity $v^*$ is proved as a linear combination of $v_1$ and $v_2$,

$$v^* = \mu_1^v v_2 + \mu_2^v v_1 \,, \tag{1.2.1}$$

where the mixing coefficient $\mu_k^v$ satisfies

$$\mu_k^v(\beta_k, d_k^v) = \frac{\beta_k}{\beta_k + d_k^v \beta_{k'}} \tag{1.2.2}$$

and $d_k^v$ is defined in (1.1.8). The property $d_1^v d_2^v = 1$ from (1.1.8) imposes the fact that $\mu_1^v + \mu_2^v = 1$. The boundary data

$$\mu_k^v\big|_{\beta_k=0} = 0\,, \quad \mu_k^v\big|_{\beta_k=1} = 1 \tag{1.2.3}$$

implies that $v^* = v_k$ in the limit of vanishing $\beta_k$. We assume that $\mu_k^v \geq 0$ and that $\mu_k^v/\beta_k$ is continuous on $0 \leq \beta_k \leq 1$ and for all $t$. Therefore $\mu_k^v(\beta_k, d_k^v)$ is a $C^\infty$ function of $\beta_k$ and $d_k^v$.

The identity (1.2.1) gives a zero parameter model, with the velocity constitutive law $d_k^v$ given implicitly in terms of the solution. The coefficient $d_k^v$, defined in (1.1.8), can be viewed as a ratio of volume creation terms for the two multiphase species. We assume as a closure relation, that the ratio is spatially uniform, namely

$$\frac{\partial d_k^v}{\partial z} = 0\,. \tag{1.2.4}$$

This condition states that the relative extent of volume creation for the two fluid species is independent of the spatial location in the mixing zone. Thus we assume that $d_k^v = d_k^v(t)$ is a function of $t$ alone. We reformulate (1.1.8) as follows

$$\begin{aligned} d_k^v &= \frac{\displaystyle\int_{Z_k}^{Z_{k'}} \frac{\partial v_{k'}}{\partial z} + \frac{1}{\rho_{k'}}\frac{D_{k'}\rho_{k'}}{Dt}\, dz}{\displaystyle\int_{Z_k}^{Z_{k'}} \frac{\partial v_k}{\partial z} + \frac{1}{\rho_k}\frac{D_k\rho_k}{Dt}\, dz} = \frac{V_{k'} - v_{k'}(Z_k, t) + \displaystyle\int_{Z_k}^{Z_{k'}} \frac{1}{\rho_{k'}}\frac{D_{k'}\rho_{k'}}{Dt}\, dz}{-V_k + v_k(Z_{k'}, t) + \displaystyle\int_{Z_k}^{Z_{k'}} \frac{1}{\rho_k}\frac{D_k\rho_k}{Dt}\, dz} \\ &= \frac{V_{k'} - v_k(Z_{k'}, t) - \displaystyle\int_{Z_k}^{Z_{k'}} \beta_k \left(\frac{1}{\rho_k}\frac{D_k\rho_k}{Dt} - \frac{1}{\rho_{k'}}\frac{D_{k'}\rho_{k'}}{Dt}\right) dz}{-V_k + v_{k'}(Z_k, t) + \displaystyle\int_{Z_k}^{Z_{k'}} \beta_{k'} \left(\frac{1}{\rho_k}\frac{D_k\rho_k}{Dt} - \frac{1}{\rho_{k'}}\frac{D_{k'}\rho_{k'}}{Dt}\right) dz} \end{aligned} \tag{1.2.5}$$

by using (1.2.4) and the identity

$$\begin{aligned} v_k(Z_{k'}, t) - v_{k'}(Z_k, t) &= \int_{Z_k}^{Z_{k'}} \frac{\partial}{\partial z} (\beta_k v_k + \beta_{k'} v_{k'})\, dz \\ &= -\int_{Z_k}^{Z_{k'}} \frac{\beta_k}{\rho_k} \frac{D_k \rho_k}{Dt} + \frac{\beta_{k'}}{\rho_{k'}} \frac{D_{k'} \rho_{k'}}{Dt}\, dz \end{aligned} \tag{1.2.6}$$

from (1.1.3). This form of closure is physically reasonable and experimentally testable. For each order of perturbation theory examined here, the relation (1.2.5) enters into the equations for the solution variables of the same order, and supplies an otherwise missing constraint on the pressures, assuming the constitutive law $d_k^v$ is known, or specifies the constitutive law if a specific resolution of the nonuniqueness of the order by order pressure has been selected. In the present context the expansions are examined only through second order, which is the first order in which the pressure makes zero contribution to (1.2.5) and to the constitutive law $d_k^v$.

A model for the interface pressure $p^*$ is given as a similar expression to (1.2.1) when the flow is assumed weakly compressible,

$$p^* = \mu_1^p p_2 + \mu_2^p p_1 \,, \tag{1.2.7}$$

where $\mu_k^p \geq 0$ and $\mu_1^p + \mu_2^p = 1$. Consistency of $p^*$ with the microphysical equations requires that $p^* = p_k$ in the limit of vanishing $\beta_k$, which translates into boundary conditions

$$\mu_k^p\big|_{\beta_k=0} = 0\,, \quad \mu_k^p\big|_{\beta_k=1} = 1\,. \tag{1.2.8}$$

The coefficient $\mu_k^p$ is assumed to depend on spatially dimensionless quantities only. On the basis of the boundary conditions (1.2.8) on $\mu_k^p$ and the freedom to choose a common scaling factor for both the numerator and denominator, we can restrict the fractional linear model to

$$\mu_k^p(\beta_k, d_k^p) = \frac{\beta_k}{\beta_k + d_k^p(t)\beta_{k'}}\,. \tag{1.2.9}$$

It is assumed that $\mu_k^p/\beta_k$ is continuous on $0 \leq \beta_k \leq 1$ and for all $t$. Thus $\mu_k^q(\beta_k, d_k^q)$ is a $C^\infty$ function of $\beta_k$ and $d_k^p$. Imposing the condition $\mu_1^p + \mu_2^p = 1$ leads to the requirement that $d_1^p d_2^p = 1$. Thus the closure $p^*$ contains one free time-dependent function, namely $d_1^p$ (or $d_2^p$).

### 1.3 Incompressible Flow

The equations of incompressible fluid flow for volume fraction $\beta_k^\infty$, velocity $v_k^\infty$, and scalar pressure $p_k^\infty$ are given by

$$\frac{\partial \beta_k^\infty}{\partial t} + v^{*\infty}\frac{\partial \beta_k^\infty}{\partial z} = 0\,, \tag{1.3.1}$$

$$\beta_k^\infty \frac{\partial v_k^\infty}{\partial z} + \left(v_k^\infty - v^{*\infty}\right)\frac{\partial \beta_k^\infty}{\partial z} = 0\,, \tag{1.3.2}$$

$$\begin{aligned}\beta_k^\infty \rho_k^\infty \left(\frac{\partial v_k^\infty}{\partial t} + v_k^\infty \frac{\partial v_k^\infty}{\partial z}\right) &+ \beta_k^\infty \frac{\partial p_k^\infty}{\partial z} \\ &+ \left(p_k^\infty - p^{*\infty}\right)\frac{\partial \beta_k^\infty}{\partial z} = \beta_k^\infty \rho_k^\infty g,\end{aligned} \tag{1.3.3}$$

together with the constraint

$$\beta_1^\infty + \beta_2^\infty = 1\,, \tag{1.3.4}$$

where $\rho_k^\infty$ is the constant density of phase $k$. Initial data for (1.3.1)-(1.3.4)

$$\left(\beta_k^\infty(z,0),\ v_k^\infty(z,0),\ p_k^\infty(z,0)\right) = \left(\beta_{k,0}^\infty(z),\ v_{k,0}^\infty(z),\ p_{k,0}^\infty(z)\right) \tag{1.3.5}$$

are assumed to be piecewise $C^1$-smooth and $\beta_{k,0}^\infty(z)$ is assumed to be monotone. Under the assumption $\rho_1^\infty < \rho_2^\infty$, we consider a mixing layer occupying a planar strip $Z_1^\infty(t) < z < Z_2^\infty(t)$ with the fluid below the strip purely phase 2 (heavy) and above the strip purely phase 1 (light) and the fluid far from the mixing layer at rest, where $Z_k^\infty = Z_k^\infty(t)$ denotes the position of the mixing zone edge $k$, defined as the location of vanishing $\beta_k^\infty(z,t)$. We assume $V_1^\infty < 0 < V_2^\infty$, where $V_k^\infty = \dot{Z}_k^\infty$ is the velocity of edge $k$. We also assume that $\beta_k^\infty$ is smooth and monotone for all $t$, $Z_1^\infty(t) < z < Z_2^\infty(t)$, and that $v_1^\infty$ ($v_2^\infty$) is continuous across the upper (lower) mixing zone edge, where $\beta_2^\infty = 0$ ($\beta_1^\infty = 0$). At edge $k$, the following boundary conditions must hold,

$$\beta_k^\infty = 0\,, \quad v_k^\infty = V_k^\infty \quad \text{at } z = Z_k^\infty(t)\,. \tag{1.3.6}$$

In [16] an exact identity for $v^{*\infty}$ is

$$\begin{aligned} v^{*\infty} &= \frac{\beta_1^\infty(\partial v_1^\infty/\partial z)v_2^\infty + \beta_2^\infty(\partial v_2^\infty/\partial z)v_1^\infty}{\beta_1^\infty(\partial v_1^\infty/\partial z) + \beta_2^\infty(\partial v_2^\infty/\partial z)} \\ &\equiv \mu_1^{v\infty} v_2^\infty + \mu_2^{v\infty} v_1^\infty \end{aligned} \tag{1.3.7}$$

which follows from (1.3.1) and (1.3.2). Here the mixing coefficient $\mu_k^{v\infty}$ is defined as the fractional linear form,

$$\mu_k^{v\infty}(\beta_k^\infty, d_k^{v\infty}) = \frac{\beta_k^\infty}{\beta_k^\infty + d_k^{v\infty}\beta_{k'}^\infty} \tag{1.3.8}$$

and the coefficient

$$d_k^{v\infty} = \frac{\partial v_{k'}^\infty/\partial z}{\partial v_k^\infty/\partial z} \tag{1.3.9}$$

measures the relative extent of volume creation of multiphase volume for the two species. The assumption that the ratio (1.3.9) of velocity divergences is constant in space, namely

$$\frac{\partial d_k^{v\infty}}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial v_{k'}^\infty/\partial z}{\partial v_k^\infty/\partial z}\right) = 0 \tag{1.3.10}$$

is equivalent to the closure relation

$$d_k^{v\infty} = \frac{|V_{k'}^\infty(t)|}{|V_k^\infty(t)|}\,. \tag{1.3.11}$$

In fact, (1.3.11) enters to the zero-th order term of the volume creation ratio (1.2.5) for two compressible fluid species. Thus we require as a closure assumption, that $d_k^{v\infty} = d_k^{v\infty}(t)$ is a function of $t$ alone.

A similar expression for the average quantity $p^{*\infty}$ is modeled as a convex linear combination of $p_1^\infty$ and $p_2^\infty$,

$$p^{*\infty} \;=\; \mu_1^{p\infty} p_2^\infty + \mu_2^{p\infty} p_1^\infty\,, \tag{1.3.12}$$

where $\mu_k^{p\infty} \geq 0$, $\mu_1^{p\infty} + \mu_2^{p\infty} = 1$. The fractional linear model for $\mu_k^{p\infty}$,

$$\mu_k^{p\infty}(\beta_k^\infty, d_k^{p\infty}) = \frac{\beta_k^\infty}{\beta_k^\infty + d_k^{p\infty}(t)\beta_{k'}^\infty} \tag{1.3.13}$$

has been proposed in [10] by assuming that the mixing coefficient $\mu_k^{p\infty}$ depends only on spatially dimensionless quantities. The constitutive coefficient $d_k^{p\infty}$ is modeled by

$$d_k^{p\infty} = \frac{\rho_{k'}^\infty}{\rho_k^\infty}\,. \tag{1.3.14}$$

Thus the closure $q^{*\infty}$, $q = v$, $p$, contains no free parameters.

## 1.4 Analytic Solutions for Incompressible Flow

Analytic solutions of the incompressible problem (1.3.1)-(1.3.5) have been obtained in $z \in R$, $t \geq 0$ and are piecewise $C^1$ functions with discontinuous derivatives at the mixing zone edges $z = Z_k^\infty(t)$ of incompressible flow. The velocities $v_k^\infty$ and $v^{*\infty}$ can be given explicitly in terms of the edge velocities $V_k^\infty$ and the volume fraction $\beta_k^\infty$. The volume fraction $\beta_k^\infty$ as a function of space and time is given implicitly by a history integral of $v^{*\infty}$. In the following, we will briefly describe a solution for the volume fractions and velocities in terms of the edge trajectories for a fluid mixing layer.

The incompressible velocities satisfy (1.3.2) within the mixing zone $Z_1^\infty < z < Z_2^\infty$ and satisfy

$$\frac{\partial v_k^\infty}{\partial z} = 0 \tag{1.4.1}$$

outside of the mixing zone, $(-1)^{k'} z > (-1)^{k'} Z_{k'}^\infty$. The solution to the ODE (1.4.1) which satisfies the boundary condition that $v_1^\infty$ vanishes at the upper wall of the finite but large domain is

$$v_1^\infty = v_1^\infty(Z_2^\infty, t) = 0 \quad \text{in } z \geq Z_2^\infty . \tag{1.4.2}$$

Next we solve (1.3.2) for $v_k^\infty$ in $Z_1^\infty < z < Z_2^\infty$. From (1.3.2), simple calculations show

$$\frac{\partial}{\partial z}\left(\beta_k^\infty v_k^\infty + \beta_{k'}^\infty v_{k'}^\infty\right) = 0 \,, \tag{1.4.3}$$

$$\frac{\partial}{\partial z}\left(v_{k'}^\infty - d_k^{v\infty} v_k^\infty\right) = 0 \,. \tag{1.4.4}$$

These ODE's can be integrated to yield

$$\begin{aligned} \beta_k^\infty v_k^\infty + \beta_{k'}^\infty v_{k'}^\infty &= U_0(t) \,, & (1.4.5)\\ v_{k'}^\infty - d_k^{v\infty} v_k^\infty &= U_{1,k}(t) & (1.4.6) \end{aligned}$$

which are purely functions of $t$. Evaluation at the mixing zone boundaries $Z_k^\infty(t)$ yields

$$\begin{aligned} U_0(t) &\equiv v_k^\infty(Z_{k'}^\infty, t) = v_{k'}^\infty(Z_k^\infty, t) \,, & (1.4.7)\\ U_{1,k}(t) &\equiv v_{k'}^\infty(Z_k^\infty, t) - d_k^{v\infty} v_k^\infty(Z_k^\infty, t) & \\ &= v_{k'}^\infty(Z_{k'}^\infty, t) - d_k^{v\infty} v_k^\infty(Z_{k'}^\infty, t) \,. & (1.4.8) \end{aligned}$$

The mean velocity $\beta_k^\infty v_k^\infty + \beta_{k'}^\infty v_{k'}^\infty$ in (1.4.5) must join continuously to $v_1^\infty$ at $z = Z_2^\infty$ and to $v_2^\infty$ at $z = Z_1^\infty$, the former of which has been determined in (1.4.2). Thus we obtain

$$U_0(t) = v_2^\infty(Z_1^\infty, t) = v_1^\infty(Z_2^\infty, t) = 0\,. \tag{1.4.9}$$

The identity (1.4.8) leads to the relation

$$d_k^{v\infty}(t) = -\frac{v_{k'}^\infty(Z_{k'}^\infty, t)}{v_k^\infty(Z_k^\infty, t)} = \frac{|V_{k'}^\infty|}{|V_k^\infty|} \tag{1.4.10}$$

by use of (1.3.6), proving (1.3.11). Substituting (1.4.6) for $v_{k'}^\infty$ into (1.4.5), the solution for the incompressible velocity is given by

$$v_k^\infty(\beta_k^\infty, t) = \frac{V_k^\infty |V_{k'}^\infty| \beta_{k'}^\infty}{|V_k^\infty|\beta_k^\infty + |V_{k'}^\infty|\beta_{k'}^\infty} = V_k^\infty \mu_{k'}^{v\infty} \tag{1.4.11}$$

in the mixing zone. Using (1.4.9), the solution to the ODE (1.4.1) is

$$v_2^\infty = v_2^\infty(Z_1^\infty, t) = 0 \tag{1.4.12}$$

in the single phase $z \le Z_1^\infty$.

Solving the interface equation (1.3.1) by the method of characteristics gives an implicit equation for the volume fraction profile,

$$z(\beta_k^\infty, t) = z(\beta_k^\infty, 0) + \int_0^t v^{*\infty}(\beta_k^\infty, s)ds\,. \tag{1.4.13}$$

If the initial mixing layer is negligibly thin, then differentiating (1.4.13), we obtain the equation

$$\frac{\partial z}{\partial \beta_k^\infty} = (-1)^{k'} 2 \int_0^t \frac{V_{k'}^{\infty\,2}(s) V_k^{\infty\,2}(s)}{\left(|V_k^\infty(s)|\beta_k^\infty + |V_{k'}^\infty(s)|\beta_{k'}^\infty\right)^3} ds\,. \tag{1.4.14}$$

All of these results have been presented in great detail in [9, 11].

## 1.5 Nondimensionalization of Compressible Flow

The first step in understanding the singular limit of incompressible flow is through the nondimensionalization of the compressible fluid equations. The equations of compressible isentropic ideal fluid flow are written in nondimensional form in terms of $(\beta_k^\lambda, \rho_k^\lambda, v_k^\lambda)$ depending on a large nondimensional parameter $\lambda$. Since the equation of state disappears from the equations in this limit, we assume a simple equation of state for the compressible equations, that of a $\gamma$-law isentropic gas,

$$p(\rho, \lambda) \equiv \lambda^2 p(\rho) = \lambda^2 A\rho^\gamma = A_\lambda \rho^\gamma\,, \quad \gamma > 1\,, \tag{1.5.1}$$

where $A$ is the entropy of the fluid. An isentropic gas does not have temperature, energy, or entropy as a dynamic variable, but has a (constant) entropy $A_\lambda = \lambda^2 A$ which becomes large as $\lambda$ increases. The above formula defines $p(\rho)$ in terms of $p(\rho,\lambda)$: $p(\rho) = \lambda^{-2} p(\rho,\lambda)$. The pressure $p(\rho)$ is bounded as $\lambda \to \infty$. Here $p(\rho,\lambda)$ is a one-parameter family of equations of state with $\frac{dp(\rho,\lambda)}{d\rho} > 0$ for $\rho > 0$ and $p(\rho,\lambda) \to \infty$ as $\lambda \to \infty$. The sound speed calculated from (1.5.1) is

$$c \equiv \left(\frac{dp(\rho^\lambda,\lambda)}{d\rho^\lambda}\right)^{1/2} = \left(\gamma p(\rho^\lambda,\lambda)/\rho^\lambda\right)^{1/2} = \lambda\left(\frac{dp(\rho^\lambda)}{d\rho^\lambda}\right)^{1/2}.$$

In particular, the Mach number $M$ is defined by the ratio of the typical fluid speed $|v_m|$ to the typical sound speed $c_m$,

$$M = \frac{|v_m|}{c_m} = \frac{|v_m|}{\left(\gamma p(\rho_m,\lambda)/\rho_m\right)^{1/2}}.$$

In this physical model, entropy and temperature increase while velocity and density are fixed. Therefore $c \to \infty$ and $M \to 0$. On the other hand, a $\lambda$-dependent change of length scales would keep the global system entropy and the temperature fixed and send all $v^\lambda$'s to zero when a length scale tends to zero as $\lambda \to \infty$. Also pressure, which is force per unit area, would remain fixed while the extensive quantity, force, tends to zero due to the change in units for measurement of area. Then $c$ is fixed and also $M \to 0$. This is the normal version of the incompressible limit. Introducing the pressure $p^\lambda = p(\rho^\lambda)$, with

$$\rho(p) = \left(A^{-1}p\right)^{1/\gamma}, \quad \frac{dp(\rho)}{d\rho} = \gamma A^{1/\gamma} p^{(\gamma-1)/\gamma},$$

we obtain the following hyperbolic system of the dimensionless compressible equations:

$$\frac{\partial \beta_k^\lambda}{\partial t} + v^{*\lambda}\frac{\partial \beta_k^\lambda}{\partial z} = 0\,, \tag{1.5.2}$$

$$\beta_k^\lambda\left(\frac{\partial \rho_k^\lambda}{\partial t} + v_k^\lambda \frac{\partial \rho_k^\lambda}{\partial z}\right) + \beta_k^\lambda \rho_k^\lambda \frac{\partial v_k^\lambda}{\partial z} + \rho_k^\lambda (v_k^\lambda - v^{*\lambda})\frac{\partial \beta_k^\lambda}{\partial z} = 0\,, \tag{1.5.3}$$

$$\begin{aligned}\beta_k^\lambda \rho_k^\lambda \left( \frac{\partial v_k^\lambda}{\partial t} + v_k^\lambda \frac{\partial v_k^\lambda}{\partial z} \right) + \lambda^2 \beta_k^\lambda \frac{\partial p_k^\lambda}{\partial z} &+ \lambda^2 (p_k^\lambda - p^{*\lambda}) \frac{\partial \beta_k^\lambda}{\partial z} \\ &= \beta_k^\lambda \rho_k^\lambda g\,, \qquad (1.5.4)\\ \beta_1^\lambda + \beta_2^\lambda &= 1\,. \qquad (1.5.5)\end{aligned}$$

We want to analyze the singular limit process of the solutions $\beta_k^\lambda$, $v_k^\lambda$, $\rho_k^\lambda$ and $p_k^\lambda$ as $\lambda \to \infty$.

## 2 The Asymptotic Analysis of Two-phase Flow Equations

A uniformly valid asymptotic expansion describing a singular limit process exists uniquely. Each order of asymptotic expansions for the solutions of the compressible flow (1.5.2)-(1.5.5) describing the incompressible limit process has an independent existence, defined as proportional to a derivative of the compressible solution with respect to $\lambda$ evaluated at the value $\lambda = 0$ of the expansion parameter. The slow variables and fast variables in the uniformly valid expansion, which have the slow time scale $t$ and the fast time scale $\lambda t$, $\lambda \to \infty$ exist within identical expansion orders and they are defined independently. In the exterior domain $(-1)^{k'} z > (-1)^{k'} Z_{k'}$, there exists single phase flow and the compressible flow is described by the Euler equations for $v_k$, $\rho_k$, $p_k$ with $\beta_k = 1$. Specifically, in this domain the limit process is the incompressible limit of the compressible Euler equations which has been discussed in [5, 12, 14, 15]. Main difference of our case is complications of second phase. In this paper we only discuss the outer limit process valid away from the initial curve $t = 0$ and uniformly valid in space. Further discussion on the fast variables and uniformly valid expansions of the solutions for compressible multiphase flow equations will be presented in [13].

### 2.1 Formal Expansions

Consider the dimensionless compressible isentropic ideal multi-phase equations in (1.5.2)-(1.5.5) suppressing superscript $\lambda$'s. We introduce outer limit asymptotic expansions

$$\begin{aligned}\beta_k &= \beta_k^{(0,s)} + \lambda^{-1} \beta_k^{(1,s)} + \lambda^{-2} \beta_k^{(2,s)} + O(\lambda^{-3})\,,\\ v_k &= v_k^{(0,s)} + \lambda^{-1} v_k^{(1,s)} + \lambda^{-2} v_k^{(2,s)} + O(\lambda^{-3})\,, \qquad (2.1.1)\\ \rho_k &= \rho_k^{(0,s)} + \lambda^{-1} \rho_k^{(1,s)} + \lambda^{-2} \rho_k^{(2,s)} + O(\lambda^{-3})\,.\end{aligned}$$

The equation of state gives us the expansion

$$
\begin{aligned}
p_k &= p_k^{(0,s)} + \lambda^{-1} p_k^{(1,s)} + \lambda^{-2} p_k^{(2,s)} + O(\lambda^{-3}) \qquad (2.1.2)\\
&= p_k(\rho_k^{(0,s)}) + \lambda^{-1} c_k^2(\rho_k^{(0,s)})\rho_k^{(1,s)} \\
&\quad + \lambda^{-2}\left(\frac{1}{2}\frac{d^2 p_k}{d\rho_k^2}(\rho_k^{(0,s)})\rho_k^{(1,s)2} + c_k^2(\rho_k^{(0,s)})\rho_k^{(2,s)}\right) + O(\lambda^{-3}),
\end{aligned}
$$

where $c_k^2(\rho) = dp_k/d\rho_k(\rho) = \gamma_k A_k \rho^{\gamma_k - 1}$. The variables with a slow time scale in asymptotic expansions of the solutions of (1.5.2)-(1.5.5) are determined in the mixing zone $Z_1 \le z \le Z_2$ and in the single phase region $(-1)^{k'} z \ge (-1)^{k'} Z_{k'}$, respectively. The equations for the slow variables are derived by repeated application of outer limit to the compressible equations (1.5.2)-(1.5.5) and by equating terms of the same order of $\lambda$. The leading order terms satisfy nonlinear differential equations. The slow variables of higher order in $\lambda^{-1}$ satisfy simple differential equations which are linearized incompressible equations. The variables $\beta_k^{(m,s)}$ and $v_k^{(m,s)}$, $m = 0, 1, 2$, solve a subsystem of equations. The remaining equations can thus be viewed as equations for $p_k^{(m,s)}$ alone. An effective velocity is introduced to decouple $v_k^{(m,s)}$ and $\beta_k^{(m,s)}$. Decoupling is a part of solvability. Our analyses of $p_k^{(0,s)}$, $p_k^{(1,s)}$ and $p_k^{(2,s)}$ use an effective pressure. We solve the equations to obtain two linear relations between the $k$ and $k'$ variables. From these relations we express the solutions in terms of the initial and the boundary data. For details, the reader may consult [13]. In higher order in $\lambda^{-1}$, there exist transition layers in the intermediate region of the mixing zone edges $Z_k$ and $Z_k^{\infty}$ for the compressible and incompressible flow. The slow variables, uniformly valid in space, are determined by matching of the outer limit and the transition-layer expansions.

## 2.2 Boundary Conditions and Asymptotic Assumptions

We specify boundary conditions for the compressible flow. Imagine a container subject to a strong downward acceleration. In the frame of the container, gravity points up. We need to keep a top on the container, so that the fluid stays inside, but the bottom can be open. This intuitive picture leads to the following set of boundary conditions. We assume existence of rigid wall at the top of a finite but large domain $\mathcal{D}$. Then the velocity is zero and the pressure is unknown there. At the bottom of this domain, we conceptually have an open container. This fixes the pressure at some ambient value, but not the velocity at the bottom of $\mathcal{D}$. This leads to the boundary conditions

$$
\begin{aligned}
v_1(z^{+\infty}, t) &= 0\,, && (2.2.1)\\
p_2(z^{-\infty}, t) &= \text{const}\,, && (2.2.2)
\end{aligned}
$$

where $z = z^{+\infty}$ ($z = z^{-\infty}$) denotes the position of the upper (lower) wall of the domain $\mathcal{D}$. The incompressible flow boundary conditions must be derived from these assumptions.

Measurement of the trajectory of the mixing zone edge must be provided as data. We assume a formal outer limit asymptotic expansion for the compressible mixing zone edge,

$$
Z_k^\lambda(t) = Z_k^{(0,s)}(t) + \lambda^{-1} Z_k^{(1,s)}(t) + \lambda^{-2} Z_k^{(2,s)}(t) + O(\lambda^{-3})\,, \qquad (2.2.3)
$$

where $Z_k^{(0,s)} = Z_k^{(0,s)}(t)$ denotes the location of vanishing $\beta_k^{(0,s)}$. Thus $Z_k^\lambda$ and each of the expansion coefficients $Z_k^{(m,s)}$, $m = 0, 1, 2$, are input to the model equations. We assume that the first leading order term equals to the incompressible edge trajectory,

$$
Z_k^{(0,s)}(t) = Z_k^{\infty}(t)\,. \qquad (2.2.4)
$$

We also assume that

$$
\begin{aligned}
(-1)^k Z_k^\lambda(t) &\geq (-1)^k Z_k^\infty(t)\,, \quad t > 0\,,\\
Z_k^\lambda(0) &= Z_k^\infty(0)
\end{aligned}
\qquad (2.2.5)
$$

and assume a similar inequality for any finite number of terms in the expansion (2.2.3). Thus,

$$
(-1)^k Z_k^{(m,s)} \geq 0\,, \quad m = 1, 2\,. \qquad (2.2.6)
$$

The edge velocity of the compressible flow satisfies $V_k = \dot{Z}_k = v_k(Z_k, t)$ and therefore, it must have an asymptotic expansion associated with the expansion (2.2.3) in the form

$$
V_k^\lambda(t) = V_k^{(0,s)}(t) + \lambda^{-1} V_k^{(1,s)}(t) + \lambda^{-2} V_k^{(2,s)}(t) + O(\lambda^{-3})\,. \qquad (2.2.7)
$$

From the expansion of $v_k$ in (2.1.1), we see that the leading order term of the asymptotic expansion of $V_k$ must be $v_k^{(0,s)}(Z_k^{(0,s)}, t)$, where $Z_k^{(0,s)} = Z_k^{(0,s)}(t)$

denotes the location of vanishing $\beta_k^{(0,s)}$. Thus $V_k^{(0,s)}(t) = v_k^{(0,s)}(Z_k^{(0,s)}, t)$. From (2.2.3) and (2.2.7), we note that

$$\frac{dZ_k^{(m,s)}(t)}{dt} = V_k^{(m,s)}(t), \quad m = 0, 1, 2. \tag{2.2.8}$$

In Sec. 1.1 the mixing coefficient $\mu_k^q$, $q = v, p$, is given as the fractional linear form, (1.2.2) and (1.2.9), with the constitutive law $d_k^q$. An asymptotic expansion for $\mu_k^q$ can be directly derived from the expansions of $\beta_k$, introduced in (2.1.1), and $d_k^q$. We assume that the coefficients $d_k^v(t)$ and $d_k^p(t)$ have a formal outer limit asymptotic expansion

$$d_k^q(t, \lambda) = d_k^{q(0,s)}(t) + \lambda^{-1} d_k^{q(1,s)}(t) + \lambda^{-2} d_k^{q(2,s)}(t) + O(\lambda^{-3}), \tag{2.2.9}$$

where $q = v, p$,

$$d_k^{v(0,s)}(t) = \frac{\left|V_{k'}^{(0,s)}\right|}{\left|V_k^{(0,s)}\right|} = \frac{\left|v_{k'}^{(0,s)}(Z_{k'}^{(0,s)}, t)\right|}{\left|v_k^{(0,s)}(Z_k^{(0,s)}, t)\right|}, \tag{2.2.10}$$

$$d_k^{p(0,s)}(t) = \frac{\rho_{k'}^{(0,s)}(Z_{k'}^{(0,s)}, t)}{\rho_k^{(0,s)}(Z_k^{(0,s)}, t)}. \tag{2.2.11}$$

We also assume that $d_k^{q(j,s)}(t)$, $j = 0, 1, 2$, belongs to $C^1$ in $0 < t < \infty$. Specifically, we assume that for $0 < t \leq T$,

$$\left\|d_k^{q(j,s)}\right\| + \left\|\frac{dd_k^{q(j,s)}}{dt}\right\| \leq C_1(T), \quad j = 0, 1, 2, \tag{2.2.12}$$

where $||\cdot||$ is the maximum norm and $C_1(T)$ is a constant depending only on $T$, for any $T, 0 < T < \infty$. The property $d_1^q d_2^q = 1$, $q = v, p$, gives the relations

$$d_1^{q(0,s)} d_2^{q(0,s)} = 1, \tag{2.2.13}$$

$$d_1^{q(0,s)} d_2^{q(1,s)} + d_1^{q(1,s)} d_2^{q(0,s)} = 0, \tag{2.2.14}$$

$$d_1^{q(0,s)} d_2^{q(2,s)} + d_1^{q(1,s)} d_2^{q(1,s)} + d_1^{q(2,s)} d_2^{q(0,s)} = 0 \tag{2.2.15}$$

between the terms in the expansion (2.2.9).

We assume

$$d_k^{v\infty}(t) \neq d_k^{p\infty} \quad \text{for all } t. \tag{2.2.16}$$

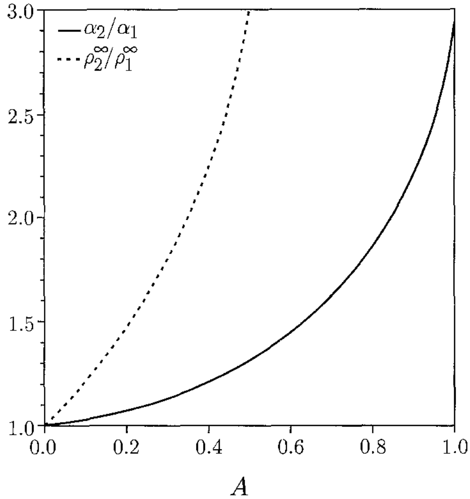


Figure 1: The expansion ratio $\alpha_2/\alpha_1$ and the density ratio $\rho_2^\infty/\rho_1^\infty$ of the mixing zone as a function of the Atwood ratio $A = (\rho_2^\infty - \rho_1^\infty)/(\rho_2^\infty + \rho_1^\infty)$.

This assumption is used in the proof of the second order term $p_k^{(2,s)}$. In the special case of RT mixing of the incompressible self-similar flow under a constant acceleration $g > 0$, $Z_k^\infty(t) = (-1)^k \alpha_k A g t^2$, where $\alpha_1$ and $\alpha_2$ are positive constants which depend on the Atwood ratio $A = (\rho_2^\infty - \rho_1^\infty)/(\rho_2^\infty + \rho_1^\infty)$. The ratio $|V_2^\infty/V_1^\infty| = \alpha_2/\alpha_1 = d_1^{v\infty}$ and as a constitutive assumption $\rho_2^\infty/\rho_1^\infty = d_1^{p\infty}$ are constants, so $d_k^{v\infty}(t) \neq d_k^{p\infty}$ for all Atwood numbers $A \neq 0$ in this problem. See Figure 1.

### 2.3 The Slow Transition Layers

In this section, we discuss transition layers in the gap of the mixing zone edges $Z_k$ and $Z_k^\infty$, caused by moving of the compressible edge faster than the mixing zone edge of the incompressible flow. Since we assume (2.2.6) in the outer limit expansion for $Z_k$, there are two new transitional regions

$$z \in \left[(-1)^i Z_i^\infty,\ (-1)^i \left(Z_i^\infty + \lambda^{-1} Z_i^{(1,s)}\right)\right], \quad i = k, k',$$

in the first order expansion, and similar additional regions at every new order. The slow variables are matched continuously order by order at the boundaries of these layers. Solution of the asymptotic transitional terms is needed to provide boundary conditions for the slow variables by matching. In this way a uniformly valid expansion in $z$ is defined for the slow variables.

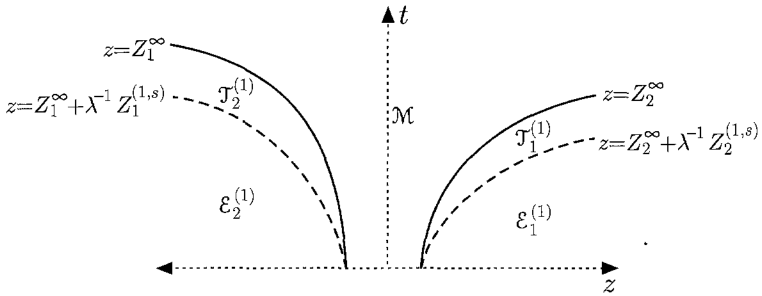


Figure 2: The five layers in the first order expansion

The first order slow variables have the transition layers through $z = Z_i^\infty + \lambda^{-1} Z_i^{(1,s)}, i = k, k'$. Therefore five regions, the lower exterior, lower transitional, incompressible mixing zone, upper transitional and upper exterior define the first order expansion. As seen in Figure 2, we define the five regions

$$
\begin{aligned}
\mathcal{E}_{k'}^{(1)} &= \left\{(z,t) : (-1)^k \left(Z_k^\infty + \lambda^{-1} Z_k^{(1,s)}\right) \le (-1)^k z\right\}, \\
\mathcal{T}_{k'}^{(1)} &= \left\{(z,t) : (-1)^k Z_k^\infty \le (-1)^k z < (-1)^k \left(Z_k^\infty + \lambda^{-1} Z_k^{(1,s)}\right)\right\}, \\
\mathcal{M} &= \left\{(z,t) : Z_1^\infty < z < Z_2^\infty\right\}.
\end{aligned}
\tag{2.3.1}
$$

We introduce a new inner space variable

$$
\zeta_i^{(1)} = \lambda\left[z - \left(Z_i^\infty + \lambda^{-1} Z_i^{(1,s)}\right)\right], \quad i = k, k' \tag{2.3.2}
$$

and assume transition layer expansions of the form

$$
\begin{aligned}
\beta_k &= \beta_k^{(0,st)}(\zeta_i^{(1)}, t) + \lambda^{-1}\beta_k^{(1,st)}(\zeta_i^{(1)}, t) + \lambda^{-2}\beta_k^{(2,st)}(\zeta_i^{(1)}, t) + \cdots, \\
v_k &= v_k^{(0,st)}(\zeta_i^{(1)}, t) + \lambda^{-1} v_k^{(1,st)}(\zeta_i^{(1)}, t) + \lambda^{-2} v_k^{(2,st)}(\zeta_i^{(1)}, t) + \cdots, \\
\rho_k &= \rho_k^{(0,st)}(\zeta_i^{(1)}, t) + \lambda^{-1}\rho_k^{(1,st)}(\zeta_i^{(1)}, t) + \lambda^{-2}\rho_k^{(2,st)}(\zeta_i^{(1)}, t) + \cdots, \\
p_k &= p_k^{(0,st)}(\zeta_i^{(1)}, t) + \lambda^{-1} p_k^{(1,st)}(\zeta_i^{(1)}, t) + \lambda^{-2} p_k^{(2,st)}(\zeta_i^{(1)}, t) + \cdots
\end{aligned}
\tag{2.3.3}
$$

in $(-1)^{i+1} Z_i^{(1,s)} \le (-1)^i \zeta_i^{(1)} \le 0$. We make the change of variables from $(z,t)$

to $(\zeta_i^{(1)}, t)$. Eqs. (1.5.2)-(1.5.5) become

$$\frac{\partial \beta_k}{\partial t} - \lambda\left(V_i^{\infty} + \lambda^{-1} V_i^{(1,s)}\right)\frac{\partial \beta_k}{\partial \zeta_i^{(1)}} + \lambda v^* \frac{\partial \beta_k}{\partial \zeta_i^{(1)}} = 0\,, \tag{2.3.4}$$

$$\beta_k\left(\frac{\partial \rho_k}{\partial t} - \lambda\left(V_i^{\infty} + \lambda^{-1} V_i^{(1,s)}\right)\frac{\partial \rho_k}{\partial \zeta_i^{(1)}} + \lambda v_k \frac{\partial \rho_k}{\partial \zeta_i^{(1)}}\right) + \lambda \beta_k \rho_k \frac{\partial v_k}{\partial \zeta_i^{(1)}}$$
$$+\lambda \rho_k (v_k - v^*) \frac{\partial \beta_k}{\partial \zeta_i^{(1)}} = 0\,, \tag{2.3.5}$$

$$\beta_k \rho_k\left(\frac{\partial v_k}{\partial t} - \lambda\left(V_i^{\infty} + \lambda^{-1} V_i^{(1,s)}\right)\frac{\partial v_k}{\partial \zeta_i^{(1)}} + \lambda v_k \frac{\partial v_k}{\partial \zeta_i^{(1)}}\right) + \lambda^3 \beta_k \frac{\partial p_k}{\partial \zeta_i^{(1)}}$$
$$+\lambda^3 (p_k - p^*) \frac{\partial \beta_k}{\partial \zeta_i^{(1)}} = \beta_k \rho_k g(t)\,. \tag{2.3.6}$$

We substitute the transition-layer expansions (2.3.3) into the compressible equations (2.3.4)-(2.3.6) and equate powers of $\lambda$. Since $\lambda$ is arbitrary, the coefficient of $\lambda^n$ for each order $n$ must vanish, defining differential equations for the transitional terms. The transitional variables are solved in closed form in [13]. Matching the outer limit expansions in the exterior domain with the outer edge of the transition-layer, and the inner edge of the transition layer with the outer edge of the incompressible mixing zone to $O(\lambda^{-1})$ defines the uniformly valid expansion in $z$ for the slow variables $\beta_k^{(1,s)}$, $v_k^{(1,s)}$, $\rho_k^{(1,s)}$ and $p_k^{(1,s)}$. This process also determines the zero-th order terms $v_k^{(0,s)}$, $\rho_k^{(0,s)}$ and $p_k^{(0,s)}$ uniformly in $\mathcal{T}_{k'}^{(1)}$. Therefore the zero-th order terms are defined in the region $\mathcal{E}_k^{(1)} \cup \mathcal{T}_k^{(1)} \cup \mathcal{M} \cup \mathcal{T}_{k'}^{(1)}$, extending the definition in $\mathcal{E}_k^{(1)} \cup \mathcal{T}_k^{(1)} \cup \mathcal{M}$.

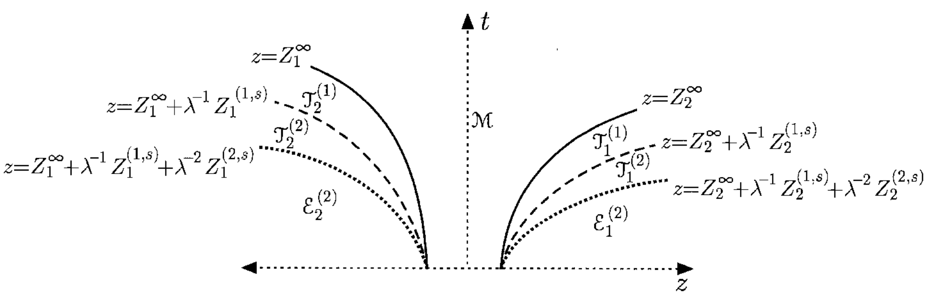


Figure 3: The seven layers in the second order expansion

In second order, there are four transition layers, the first layers defined as above and the second layers extending out to $z = Z_i^{\infty} + \lambda^{-1} Z_i^{(1,s)} + \lambda^{-2} Z_i^{(2,s)}, i = k, k'$.

Therefore, the second order slow variables have seven regions, the lower exterior, second lower transitional, first lower transitional, incompressible mixing zone, first upper transitional, second upper transitional and upper exterior. We define the seven regions as the following

$$
\begin{aligned}
\mathcal{E}_{k'}^{(2)} &= \left\{(z,t) : (-1)^k \left(Z_k^\infty + \lambda^{-1} Z_k^{(1,s)} + \lambda^{-2} Z_k^{(2,s)}\right) \geq (-1)^k z\right\}, \\
\mathcal{T}_{k'}^{(2)} &= \left\{(z,t) : (-1)^k \left(Z_k^\infty + \lambda^{-1} Z_k^{(1,s)}\right)\right. \\
&\qquad \left. \leq (-1)^k z < (-1)^k \left(Z_k^\infty + \lambda^{-1} Z_k^{(1,s)} + \lambda^{-2} Z_k^{(2,s)}\right)\right\}, \\
\mathcal{T}_{k'}^{(1)} &= \left\{(z,t) : (-1)^k Z_k^\infty \leq (-1)^k z < (-1)^k \left(Z_k^\infty + \lambda^{-1} Z_k^{(1,s)}\right)\right\}, \\
\mathcal{M} &= \left\{(z,t) : Z_1^\infty < z < Z_2^\infty\right\}.
\end{aligned}
$$

Introducing the second inner space variable

$$
\zeta_i^{(2)} = \lambda^2 \left[z - \left(Z_i^\infty + \lambda^{-1} Z_i^{(1,s)} + \lambda^{-2} Z_i^{(2,s)}\right)\right], \quad i = k, k', \tag{2.3.7}
$$

we make the change of variables from $(z,t)$ to $(\zeta_i^{(2)}, t)$ in (1.5.2)-(1.5.5), leading to the equations

$$
\frac{\partial \beta_k}{\partial t} - \lambda^2 \left(V_i^\infty + \lambda^{-1} V_i^{(1,s)} + \lambda^{-2} V_i^{(2,s)}\right) \frac{\partial \beta_k}{\partial \zeta_i^{(2)}} + \lambda^2 v^* \frac{\partial \beta_k}{\partial \zeta_i^{(2)}} = 0, \tag{2.3.8}
$$

$$
\begin{aligned}
&\beta_k \left(\frac{\partial \rho_k}{\partial t} - \lambda^2 \left(V_i^\infty + \lambda^{-1} V_i^{(1,s)} + \lambda^{-2} V_i^{(2,s)}\right) \frac{\partial \rho_k}{\partial \zeta_i^{(2)}} + \lambda v_k \frac{\partial \rho_k}{\partial \zeta_i^{(1)}}\right) \\
&\quad + \lambda^2 \beta_k \rho_k \frac{\partial v_k}{\partial \zeta_i^{(2)}} + \lambda^2 \rho_k (v_k - v^*) \frac{\partial \beta_k}{\partial \zeta_i^{(2)}} = 0,
\end{aligned} \tag{2.3.9}
$$

$$
\begin{aligned}
&\beta_k \rho_k \left(\frac{\partial v_k}{\partial t} - \lambda^2 \left(V_i^\infty + \lambda^{-1} V_i^{(1,s)} + \lambda^{-2} V_i^{(2,s)}\right) \frac{\partial v_k}{\partial \zeta_i^{(2)}} + \lambda^2 v_k \frac{\partial v_k}{\partial \zeta_i^{(2)}}\right) \\
&\quad + \lambda^4 \beta_k \frac{\partial p_k}{\partial \zeta_i^{(2)}} + \lambda^4 (p_k - p^*) \frac{\partial \beta_k}{\partial \zeta_i^{(2)}} = \beta_k \rho_k g(t).
\end{aligned} \tag{2.3.10}
$$

We substitute the second transition-layer expansions

$$
\begin{aligned}
\beta_k &= \beta_k^{(0,stt)}(\zeta_i^{(2)}, t) + \lambda^{-1} \beta_k^{(1,stt)}(\zeta_i^{(2)}, t) + \lambda^{-2} \beta_k^{(2,stt)}(\zeta_i^{(2)}, t) + \cdots, \\
v_k &= v_k^{(0,stt)}(\zeta_i^{(2)}, t) + \lambda^{-1} v_k^{(1,stt)}(\zeta_i^{(2)}, t) + \lambda^{-2} v_k^{(2,stt)}(\zeta_i^{(2)}, t) + \cdots, \\
\rho_k &= \rho_k^{(0,stt)}(\zeta_i^{(2)}, t) + \lambda^{-1} \rho_k^{(1,stt)}(\zeta_i^{(2)}, t) + \lambda^{-2} \rho_k^{(2,stt)}(\zeta_i^{(2)}, t) + \cdots, \\
p_k &= p_k^{(0,stt)}(\zeta_i^{(2)}, t) + \lambda^{-1} p_k^{(1,stt)}(\zeta_i^{(2)}, t) + \lambda^{-2} p_k^{(2,stt)}(\zeta_i^{(2)}, t) + \cdots
\end{aligned} \tag{2.3.11}
$$

into (2.3.8)-(2.3.10) and equate powers of $\lambda$. Within a single power of $\lambda$, the transitional variables of each order in $\lambda^{-1}$ are defined as a solution of simple differential equations in $(-1)^{i+1}Z_i^{(2,s)} \leq (-1)^i \zeta_i^{(2)} \leq 0$. They are determined in closed form. We match the outer limit expansions in the exterior with the outer edge of the second transition layer, the inner edge of the second transition layer with the outer edge of the first transition layer, and the inner edge of the first transition layer with the outer edge of the incompressible mixing zone continuously. Thus a uniformly valid expansion in $z$ to $O(\lambda^{-2})$ is defined for the slow variables $\beta_k^{(2,s)}$, $v_k^{(2,s)}$, $\rho_k^{(2,s)}$ and $p_k^{(2,s)}$. From this process, the zero-th order terms $v_k^{(0,s)}$, $\rho_k^{(0,s)}$ and $p_k^{(0,s)}$ and the first order terms $v_k^{(1,s)}$, $\rho_k^{(1,s)}$ and $p_k^{(1,s)}$ are also determined uniformly in $\mathcal{T}_{k'}^{(2)}$. Thus, they are defined in the region $\mathcal{E}_k^{(2)} \cup \mathcal{T}_k^{(2)} \cup \mathcal{T}_k^{(1)} \cup \mathcal{M} \cup \mathcal{T}_{k'}^{(1)} \cup \mathcal{T}_{k'}^{(2)}$, extending the definition in $\mathcal{E}_k^{(2)} \cup \mathcal{T}_k^{(2)} \cup \mathcal{T}_k^{(1)} \cup \mathcal{M} \cup \mathcal{T}_{k'}^{(1)}$. All of these estimates are given in [13].

## 2.4 Main Result

The incompressible problem (1.3.1)-(1.3.5) has piecewise $C^1$ solutions $\beta_k^\infty$, $v_k^\infty$ and $p_k^\infty$ for $z \in R$, $t \geq 0$. In [3] it was shown that degeneracy and solvability conditions always hold and therefore, there exists a one parameter family of solutions to the pressure equation (1.3.3). The resolution of this extra degree of freedom is found in the expansion of the constitutive quantity $d_k^v$, measuring the relative ratio of volume creation for the two fluids. In other words, the pressures and $d_k^v$ have linked indeterminacy: one has an arbitrary degree of freedom and once specified, the other is known. We have already seen that

$$d_1^{v\infty} = \frac{|V_2^\infty|}{|V_1^\infty|} \tag{2.4.1}$$

is a ratio of volume creation terms. In addition, we require

$$d_1^{v(1,s)} = \left(\frac{|V_2|}{|V_1|}\right)^{(1,s)} = \frac{-V_1^\infty V_2^{(1,s)} + V_2^\infty V_1^{(1,s)}}{V_1^{\infty 2}}, \tag{2.4.2}$$

$$\begin{aligned} d_1^{v(2,s)} &= \left( \frac{V_2 - v_2(Z_1,t) + \displaystyle\int_{Z_1}^{Z_2} \frac{1}{\rho_2}\frac{D_2\rho_2}{Dt}\,dz}{-V_1 + v_1(Z_2,t) + \displaystyle\int_{Z_1}^{Z_2} \frac{1}{\rho_1}\frac{D_1\rho_1}{Dt}\,dz} \right)^{(2,s)} \\ &= -\frac{d_1^{v(1,s)} V_1^{(1,s)} + d_1^{v\infty} V_1^{(2,s)} + V_2^{(2,s)}}{V_1^\infty} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{V_1^\infty}\int_{Z_1^\infty}^{Z_2^\infty}\frac{\beta_1^\infty+d_1^{v\infty}}{\rho_1^\infty c_1^2(\rho_1^\infty)}\left(\frac{\partial p_1^\infty}{\partial t}+v_1^\infty\frac{\partial p_1^\infty}{\partial z}\right)\\
&+\frac{\beta_2^\infty-1}{\rho_2^\infty c_2^2(\rho_2^\infty)}\left(\frac{\partial p_2^\infty}{\partial t}+v_2^\infty\frac{\partial p_2^\infty}{\partial z}\right)dz\\
&+\frac{1}{V_1^\infty}\frac{1+d_1^{v\infty}}{\rho_1^\infty c_1^2(\rho_1^\infty)}\int_{Z_2^\infty}^{z^{+\infty}}\frac{\partial p_1^\infty}{\partial t}dz
\end{aligned}
\tag{2.4.3}
$$

as necessary and sufficient conditions for the convergence of compressible pressures to the incompressible pressures through the second order of the expansion. Here $c_k^2(\rho_k^\infty)=\partial p_k/\partial\rho_k(\rho_k^\infty)$ and $z=z^{+\infty}$ ($z=z^{-\infty}$) is defined as the position of the upper (lower) wall of $\mathcal{D}$. These equations can be regarded as constitutive constraints on the expansion. To understand (2.4.2), we note that there are no compressible contributions to volume creation to first order in the asymptotic expansion of (1.2.5). Specifically, (2.4.3) relates the constitutive law to the selection of the incompressible pressures and it enters to the second order term in a ratio of volume creation terms for two phases in (1.2.5). Compressible flow has nonunique solutions parameterized by choice of $d_k^v$ while incompressible has nonuniqueness parameterized by pressure solutions. The identity (2.4.3) joins these and shows how one maps onto the other in incompressible limit. For self-similar flow, $d_k^v(t)=d_k^v(0)$ is independent of $t$. The initial data $p_1^\infty(Z_1^\infty(0),0)=p_2^\infty(Z_2^\infty(0),0)$ is required by the initial condition $Z_k^\infty(0)=0$. This condition imposes a solvability constraint on $d_k^{v(2,s)}$. In this case, the pressure solution exists uniquely in the incompressible limit. Refer to [13]. See Figure 4.

A fluid is in mechanical equilibrium if

$$
\rho\mathbf{F}=\nabla p\,, \tag{2.4.4}
$$

where $\rho$, $p$ and $\mathbf{F}$ are a density, a pressure and a body force per mass of a fluid. In the case of a vanishing body force and 1-dimensional space, necessary and sufficient conditions for mechanical equilibrium is that the pressure of a fluid is constant in space and time. Especially, we say that the pressures $p_k^{(1,s)}$ satisfy the $O(\lambda)$ pressure equilibrium condition if

$$
p_k^{(1,s)}=p_{k'}^{(1,s)}=p^{(1)} \tag{2.4.5}
$$

where $p^{(1)}$ is constant in space and time. In [13] it is shown that the gravity force $g=g(t)$ does not affect $p_k^{(0,s)}$, $p_k^{(1,s)}$ and the effective pressure $p_k^{(2,eff)}\equiv p_k^{(2,s)}-p_k^\infty$. Since the variation of the equation of state $p(\rho,\lambda)=\lambda^2p(\rho)$

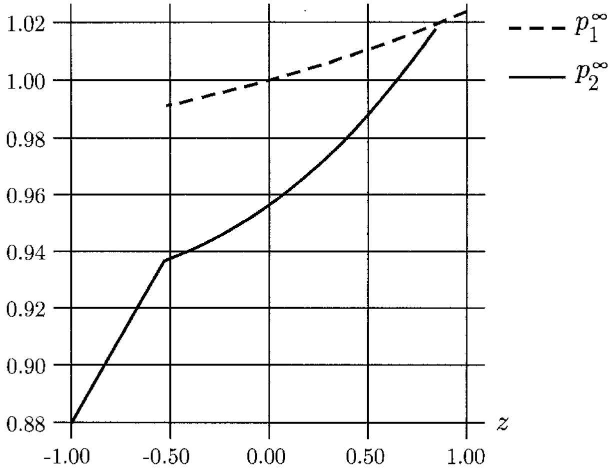


Figure 4: The incompressible pressures for self-similar flow: $t = 7.0$, $\rho_1^\infty = 0.1$, $\rho_2^\infty = 0.4$, $g = 0.3$, $\alpha_1 = 0.06$, $\alpha_2 = 0.093693$ and $Z_k^\infty = (-1)^k \alpha_k A g t^2$.

with fixed $p(\rho)$, $\partial p/\partial\rho > 0$ is given in (1.5.1), (2.4.5) is called the "$O(\lambda)$" equilibrium condition.

The volume fraction $\beta_k^\infty$, velocity $v_k^\infty$ and constant density $\rho_k^\infty$ of incompressible flow are proved to be the outer limit $\beta_k^{(0,s)}$, $v_k^{(0,s)}$ and $\rho_k^{(0,s)}$ in the expansion (2.1.1) describing this incompressible limit process. The leading order terms $p_k^{(0,s)}$ satisfy the $O(\lambda^2)$ pressure equilibrium condition defined as

$$p_k^{(0,s)} = p_{k'}^{(0,s)} = p^{(0)}, \tag{2.4.6}$$

where $p^{(0)}$ is independent of space and time. Assuming that $d_k^{v(1,s)}(t)$ in the expansion (2.2.9) satisfies (2.4.2), we derive the $O(\lambda)$ pressure equilibrium condition for $p_k^{(1,s)}$. This implies that the first order term $p_k^{(1,s)}$ in the expansion of $p_k$ is defined by the constant initial data $p_k^{(1,s)}(z,0)$ which may be assumed to be zero. In fact, it is shown that (2.4.2) is a necessary and sufficient condition for the $O(\lambda)$ pressure equilibrium condition (2.4.5) for $p_k^{(1,s)}$. The first order terms $\beta_k^{(1,s)}$ and $v_k^{(1,s)}$ are coupled. They are linear in space in the transitional regions $\mathcal{T}_i^{(1)}$ of magnitude $O(\lambda^{-1})$ and are smooth within the incompressible mixing zone. The second order term in the expansion of $p_k$ satisfies $p_k^{(2,s)} = p_k^\infty + O(\lambda^{-1})$ and therefore, it is fixed to be bounded, under the assumption that $p_k^\infty$ satisfies (2.4.3). Actually, (2.4.3) is derived as a condition equivalent to a bounded second order term $p_k^{(2,s)}$. The transition layers introduced in Sec. 2.3 affect the $O(\lambda^{-1})$ term in $p_k^{(2,s)}$. All technical issues are presented in [13]. For self-similar flow,

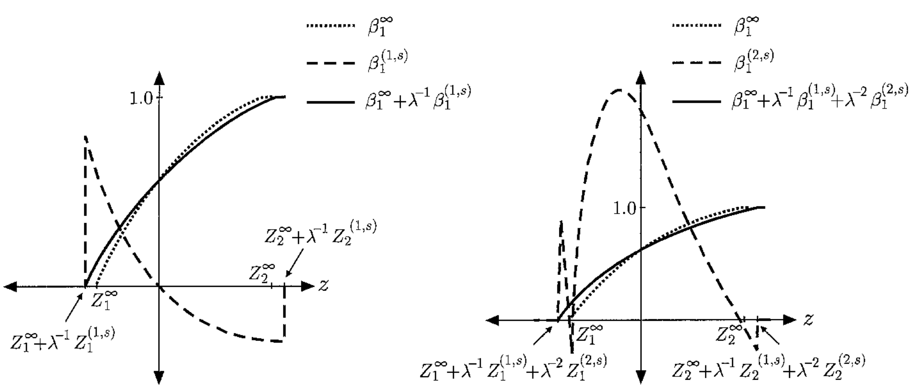


Figure 5: The volume fractions for self-similar flow: $\lambda = 10$, $Agt^2 = 8.82$, $\rho_1^\infty = 0.1$, $\rho_2^\infty = 0.4$, $g = 0.3$, $\alpha_1 = 0.06$, $\alpha_2 = 0.093693$ and $Z_k^{(2,s)} = Z_k^{(1,s)} = Z_k^\infty = (-1)^k\alpha_k Agt^2$.

see Figures 5 and 6.

## References


[1] Y. Chen, J. Glimm, D. H. Sharp and Q. Zhang. *A two-phase flow model of the Rayleigh-Taylor mixing zone*. Pys. Fluids, **8**(3): (1996), 816–825.

[2] B. Cheng, J. Glimm, X. L. Li and D. H. Sharp. *Subgrid models and DNS studies of fluid mixing*. In Proceedings of the 7th International Conference on the Pysics of Compressible Turbulent Mixing, (1999), 2000. submitted.

[3] B. Cheng, J. Glimm, D. Saltz and D. H. Sharp. *Boundary conditions for a two pressure two phase flow model*. Pysica D, **133:** (1999), 84–105.

[4] B. Cheng, J. Glimm and D. H. Sharp. *Density dependence of Rayleigh-Taylor and Richtmyer-Meshkov mixing fronts*. Pys. Lett. A, **268:** (2000), 366–374.

[5] D. Ebin. *Motion of slightly compressible fluids in a bounded domain I*. Comm. on Pure and Appl. Math, **35:** (1982), 451–485.

[6] J. Glimm, X.-L. Li, R. Menikoff, D. H. Sharp and Q. Zhang. *A numerical study of bubble interactions in Rayleigh-Taylor instability for compressible fluids*. Pys. Fluids A, **2**(11): (1990), 2046–2054.

[7] J. Glimm, X.-L. Li, R. Menikoff, D. H. Sharp and Q. Zhang. *Statistical theories of Rayleigh-Taylor instability for compressible fluids*. In W. Dannevik, A. Buckingham and C. Lieth, editors, *Advances in Compressible Turbulent Mixing*, pages 85–94. National Technical Information Service, U.S. Department of Commerce, 5285 Port Royal Rd. Springfield VA 22161, (1992).

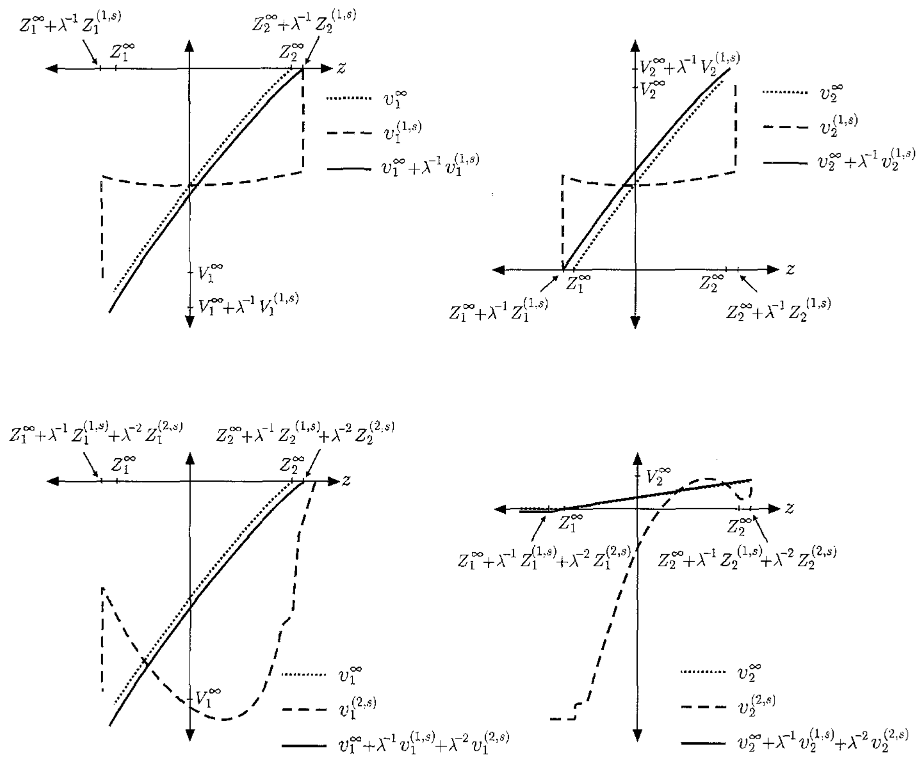

Figure 6: The velocities for self-similar flow: $\lambda = 10$, $Agt^2 = 8.82$, $\rho_1^\infty = 0.1$, $\rho_2^\infty = 0.4$, $g = 0.3$, $\alpha_1 = 0.06$, $\alpha_2 = 0.093693$ and $Z_k^{(2,s)} = Z_k^{(1,s)} = Z_k^\infty = (-1)^k \alpha_k Agt^2$.


[8] J. Glimm, D. Saltz and D. H. Sharp. *A general closure relation for incompressible mixing layers induced by interface instabilites.* In G. Jourdan and L. Houas, editors, Proceedings of the Sixth International Workshop on the Physics of Compressible Turbulent Mixing, pages 179–184. Imprimerie Caractère, Marseille, France, (1997).

[9] J. Glimm, D. Saltz and D. H. Sharp. *Statistical evolution of chaotic fluid mixing.* Phys. Rev. Lett., **80**(4): (1998), 712–715.

[10] J. Glimm, D. Saltz and D. H. Sharp. *Two-pressure two-phase flow.* In G.-Q. Chen, Y. Li, and X. Zhu, editors, Nonlinear Partial Differential Equations. World Scientific, Singapore, (1998).

[11] J. Glimm, D. Saltz and D. H. Sharp. *Two-phase modeling of a fluid mixing layer.* J. Fluid Mech., **378:** (1999), 119–143.

[12] J. Lorenz H.-O. Kreiss and M. J. Naughton. *Convergence of the solutions of the*

*compressible to the solutions of the incompressible navier-stokes equations*. Adv. Appl. Math, **12:** (1991), 187–214.

[13] H. Jin. *The Incompressible Limit of Compressible Multiphase Flow Equations*. PhD thesis, SUNY at Stony Brook, (2001). In Preparation.

[14] S. Klainerman and A. Majda. *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible flows*. Comm. Pure Appl. Math., **34:** (1981), 119–143.

[15] S. Klainerman and A. Majda. *Compresible and incompressible fluids*. Comm. Pure Appl. Math., **35:** (1982), 629–651.

[16] J. D. Ramshaw. Private communication.

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